Cup Anemometer Behavior in Turbulent Environments

LEIF KRISTENSEN

Risø National Laboratory, Roskilde, Denmark

(Manuscript received 5 December 1995, in final form 13 May 1996)

ABSTRACT

The behavior of the cup anemometer rotor in turbulent atmospheric flow is discussed in terms of a general equation of motion. This equates the rate of change $\dot{\tilde{s}}$ of the rotation rate $\tilde{s}$ of the rotor to a forcing $F(\tilde{s}, \tilde{h}, \tilde{w})$, which is proportional to the torque and a function of $\tilde{s}$ and of the total horizontal and the vertical wind velocity components, $\tilde{h}$, and $\tilde{w}$, respectively. To determine the so-called overspeeding, it is necessary to carry out first- and second-order perturbation calculations around the response curve obtained in a laminar flow. From this curve, which for the purpose of this paper can be considered linear, five constraints are derived between the first and second partial derivatives of F. These constraints provide sufficient information for deriving an expression for the overspeeding to which four distinctly different biases contribute—one for each of the velocity components and one from the covariance between streamwise velocity components $\tilde{u}$ and $\tilde{w}$. A phenomenological model of F in terms of the response distance ℓ, the distance constant L, a third instrument length scale Λ, and two dimensionless constants μ_1 and μ_2, defining the angular response, makes it possible to quantify the four terms in the overspeeding expression. It turns out that the most significant bias in the mean wind speed is due to lateral velocity fluctuations. Under certain conditions it may be larger than 10%. It is shown how it is possible to reduce all the other biases to acceptable levels. The bias from the lateral velocity fluctuations can also be suppressed significantly by combining the cup anemometer with a wind vane and using a signal processing technique called the vector wind-run method.

1. Introduction

Probably the most common anemometers are the rotating forms, cup, and propellers. One form or the other is often used with a wind vane for determining the *mean* of the horizontal wind velocity component. Such combinations are sturdy and reliable instrument packages. They are easy to operate and are used at weather stations, airports, wind farms, and sites where large structures, such as bridges, are under construction.

Here, we concentrate on providing a general description of the cup anemometer and its behavior in the turbulent wind, with emphasis on systematic errors in the measured mean wind speed. In this context the first thing we consider is the so-called overspeeding, which, to most people in observational meteorology, means a positive bias of the measured mean wind speed. This bias is a consequence of that property of the anemometer that responds more quickly to an increase in the wind speed than to a decrease of the same magnitude, so that the rotation rate is spending more time on the high side than on the low side of the mean. However, we shall see, not only streamwise but also lateral and vertical velocity fluctuations contribute to the mean wind bias.

There is a vast amount of literature about the cup anemometer that was invented as early as 1846 by the Irish astronomer T. R. Robinson (Middleton 1969; Wyngaard 1981) and that, in almost the same design, is still in use.

Until the end of the 1920s the main subject discussed in the literature was the linearity of the response curve, that is, the relation between the wind speed U and the angular velocity S of the cup rotor. Brazier (1914) and Patterson (1926) demonstrated that the linearity is better the shorter the diameter of the cup rotor is. Patterson (1926) determined, for a number of cup anemometers, the anemometer factor, which is defined as

$$f = \frac{U}{rS}, \tag{1}$$

where r is the radius of the cup rotor, that is, the distance from the axis to the center of one of the cups. He found that, depending on cup diameter and r the anemometer factor varies between 2.5 and 3.5. [According to Middleton (1969), the inventor Robinson tried to calibrate his instrument by mounting it on a carriage that was moved with a known speed under calm wind conditions. Based on these measurements, Robinson believed that he might have discovered a law of nature that says that f is exactly 3 in the limit where the friction in the

Corresponding author address: Dr. Leif Kristensen, The Pennsylvania State University, Dept. of Meteorology, 503 Walker Building, University Park, PA 16802.
E-mail: leif.kristensen@risoe.dk

bearings is negligible. The validity of this law was not generally accepted, and in 1872 F. Stow (Stow 1872) showed that ''the law'' $f = 3$ could not be reconciled with his experimental findings.]

In the 1920s interest turned toward understanding cup anemometer dynamics and overspeeding due to wind speed fluctuations. According to Kaganov and Yaglom (1976), the first attempt to quantify overspeeding was by Sabinin (1923), who used a simple dynamic equation of motion, with a square-wave fluctuation added to the mean wind speed as forcing. He was able to determine the bias of the mean of the cup anemometer output and, with these rather unrealistic assumptions, to calculate the exact amount of overspeeding. Through the mid-1970s, many theories and models of cup anemometer dynamics and results from experimental determinations of drag and lift forces on cups were published. Kaganov and Yaglom (1976) and Wyngaard (1981) have both given quite detailed account of this literature. One prominent contribution came from Schrenk (1929) who, according to Wyngaard (1981), was the first to suggest and determine experimentally a general, three-parameter expression for the torque on the cup rotor as a function of the wind speed and the rotation rate.

Wyngaard et al. (1974) analyzed the situation and found that none of the models of cup anemometer dynamics discussed so far were realistic enough for a satisfactory determination of overspeeding. They pointed out that a realistic model had to satisfy at least three requirements: to be nonlinear, to include proper sensitivity to both the horizontal and vertical wind components, and to be able to handle variable cup geometry. As a consequence they generalized Schrenk's (1929) approach and established a phenomenological model of the anemometer equation of motion. This model, which was quadratic in the turbulent velocity components and the output signal, contained eight empirical constants, and they showed how these constants for one particular cup anemometer could be determined in a wind tunnel. Subsequently, Kaganov and Yaglom (1976) and, independently, Busch and Kristensen (1976), were able to quantify overspeeding on basis of this model. Later, Coppin (1982) applied the technique developed by Wyngaard et al. (1974) to describe, almost to the same detail, seven different cup anemometers. Kristensen (1993) summarized the results concerning overspeeding, pointing out that there are contributions from terms that are proportional to not only the variances of the three velocity components but also to the covariance between the vertical and the streamwise velocity components. However, this last contribution, which is included by Kristensen (1993) for completeness, is, in general, small and of little practical importance.

In the following we will derive expressions for the four contributions to the overspeeding and, on the basis of experimental data, give quantitative estimates of their magnitudes. Also, we will suggest a phenomenological model for the cup anemometer forcing in terms of its physical characteristics. A list of symbols can be found in the appendix.

2. Cup-anemometer dynamics

We assume that the cup anemometer is mounted with its axis vertical. In this case, the equation for the rotation rate of the rotor can be written as

$$\dot{\tilde{s}} = F[\tilde{s}, (\tilde{u}^2 + \tilde{v}^2)^{1/2}, \tilde{w}], \quad (2)$$

where $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ are the instantaneous horizontal and vertical wind components, and $\tilde{s}$ (rad s^{-1}) is the instantaneous rotation rate of the anemometer rotor. The angular momentum of the rotor is proportional to $\tilde{s}$ and (2) just states that the rate of change of the angular momentum is equal to the net torque on the rotor. Contributing torques are caused by the wind and by the friction in the instrument bearings.

a. Calibration

The cup anemometer is usually calibrated in a wind tunnel that is operated with wind speeds in the range of interest.

In the steady state, with a constant horizontal wind speed U and a constant rotation rate S, (2) reduces to

$$F(S, U, 0) = \dot{S} = 0. \quad (3)$$

Solving (3) for S we obtain the response equation

$$S = S(U). \quad (4)$$

It is a well-known experience that the cup anemometer has a steady-state response curve that, for most purposes can be considered linear, that is, $d^2S/dU^2 = 0$. This means that we can write (4) as

$$S = (U - U_0)/\ell. \quad (5)$$

Here U_0 is a positive *offset speed.* It is no more than about 0.1 m s^{-1} for a good cup anemometer. Often it is called the *starting speed*, but this is really a misnomer; when the wind speed is very small—less than 1 m s^{-1}, say—the contribution to the total torque from the friction in the bearings becomes significant and the response expression is no longer close to being linear. The *real* starting speed will, in general, be larger than U_0.[1]

The quantity ℓ is a length scale that, when $U \gg U_0$, can be visualized as the length of the column of air that has blown through the anemometer when the rotor has

[1] It is extremely difficult to calibrate a cup anemometer when the wind speed is less than 0.5 m s^{-1}. This is because the speed always fluctuates in a wind tunnel and because very small wind speeds are difficult to determine by means of standard wind tunnel equipment (Pitot tube). It is therefore not, in general, possible to judge whether the response S to U is the same when U is increasing as when it is decreasing. In other words, whether the ''stopping'' speed is the same as the starting speed.

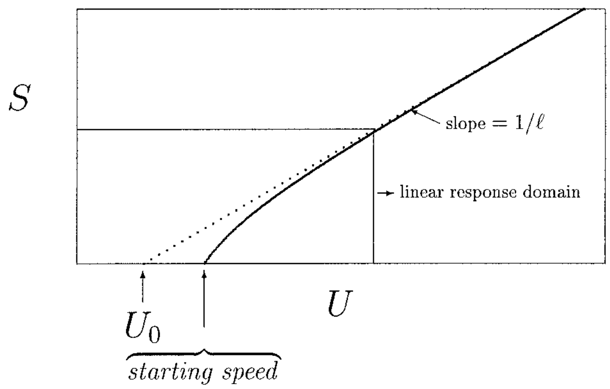

FIG. 1. The solid line represents the actual response and the dotted line is the linear expression (5). This line has the slope $1/\ell$ and intersects the U axis at U_0. The real response intersects this axis at a somewhat higher value. The response curve approaches the asymptote given by (5) in the limit $U \to \infty$. We have indicated the domain in U where the linear response is assumed to be accurate from a measuring standpoint.

turned one radian. We shall call it the *response distance*. Figure 1 is a sketch of the cup anemometer response.

b. Perturbation analysis

In operation the cup anemometer will be exposed to a wind velocity that fluctuates in magnitude and direction. We assume that the mean wind velocity is horizontal and that the fluctuations are small compared to the mean wind speed U. We decompose the instantaneous wind velocity and the anemometer response as

$$\begin{Bmatrix} \tilde{s} \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{Bmatrix} = \begin{Bmatrix} S + s \\ U + u \\ v \\ w \end{Bmatrix}, \qquad (6)$$

where S is defined by (4).

The way in which $\tilde{s}$ is decomposed does not guarantee that the average $\langle s \rangle$ is zero; in fact, $\langle s \rangle \neq 0$ is a manifestation of overspeeding.

First, we derive a first-order perturbation equation for the rotation rate in u, v, w, and s. Substituting (6) in (2) we get

$$\dot{s} = F_1' s + F_2' u + F_3' w, \qquad (7)$$

where $F_i' = \partial F/\partial\{\text{arg pos } i\}$ are taken at the point $[S(U), U, 0]$. We note that there are no terms proportional to v since

$$\begin{aligned}(\tilde{u}^2 + \tilde{v}^2)^{1/2} &= [(U + u)^2 + v^2]^{1/2} \\ &\approx U + u + v^2/(2U). \qquad (8)\end{aligned}$$

The first term containing v in the expansion of $(\tilde{u}^2 + \tilde{v}^2)^{1/2}$ is proportional to v^2 and must be included in the second-order perturbation equation.

The three coefficients F_1', F_2', and F_3' in (7) are not independent but interrelated through (3). Since, according to (4) and (5), S, is a function of U, we have

$$F_1' \frac{dS}{dU} + F_2' = \frac{1}{\ell} F_1' + F_2' = 0. \qquad (9)$$

Inspecting (7), we see that F_1' with a physical dimension of reciprocal time must be negative since perturbations cannot grow exponentially. We define the time constant as

$$\tau_0 = -\frac{1}{F_1'} \qquad (10)$$

and, substituting in (9), we get

$$F_2' = \frac{1}{(\tau_0 \ell)}. \qquad (11)$$

If the cup anemometer were equally sensitive to the perturbations u and w, then F_3' would be equal to F_2'. On the other hand, if it were ideal and not sensitive to w at all, then F_3' would be zero. We account for the deviation from the ideal response by writing

$$F_3' = \frac{\mu_1}{(\tau_0 \ell)} \qquad (12)$$

and rewrite (7) as

$$\dot{s} + \frac{s}{\tau_0} = \frac{u}{\ell \tau_0} + \mu_1 \frac{w}{\ell \tau_0}. \qquad (13)$$

If μ_1 is significantly different from zero, it will affect the operation of the cup anemometer if it is not mounted correctly with vertical axis. To see this, let us consider a situation where the mean wind has a small component W along the axis. In this case (3) must be replaced by

$$F(S, U, W) = 0. \qquad (14)$$

Since $|W| \ll U$, we can just replace $\partial S/\partial U$ by $dS/dU = 1/\ell$. Differentiating with respect to W yields

$$F_1' \frac{\partial S}{\partial W} + F_3' = 0. \qquad (15)$$

From (10) and (13) we conclude

$$\frac{\partial S}{\partial W} = \frac{\mu_1}{\ell}. \qquad (16)$$

In other words, μ_1 is a dimensionless parameter characterizing the first-order response to the vertical wind component.

Since $\partial S/\partial U$ and $\partial S/\partial W$ are independent of U, we infer

$$\frac{\partial^2 S}{\partial U^2} = \frac{\partial^2 S}{\partial U \partial W} = 0. \qquad (17)$$

However, there are no a priori reasons that the second derivative of S with respect to W is zero.

Equation (13) shows that the cup anemometer can be considered a first-order linear filter if quadratic terms in s, u, and w can be neglected. Because of the linearity it predicts that $\langle s \rangle$ is zero, that is, that there is no overspeeding. Therefore, we must expand (2) to second order in s, u, and w. The result can be written as a generalization (13) in the following way:

$$\dot{s} + \frac{s}{\tau_0} = \frac{u}{\ell\tau_0} + \mu_1\frac{w}{\ell\tau_0} + \frac{1}{\ell\tau_0}\frac{v^2}{2U} + \frac{1}{2}\{F''_{22}u^2 + 2F''_{23}uw + F''_{33}w^2\} + \frac{1}{2}\{F''_{11}s^2 + 2F''_{12}su + F''_{31}ws\}, \quad (18)$$

where $F''_{ij} = \partial^2 F/\partial(\text{arg pos } i)\partial(\text{arg pos } j)$ at the point $[S(U), U, 0]$. Again, the relation (14) provides constraints between the coefficients. There are three that are the results of differentiating (14) twice with respect to both U and W, keeping $S = S(U, W)$ a function of U and W. They are

$$F''_{11}\frac{1}{\ell^2} + 2F''_{12}\frac{1}{\ell} + F''_{22} = 0, \quad (19)$$

$$F''_{11}\frac{\mu_1}{\ell^2} + F''_{12}\frac{\mu_1}{\ell} + F''_{31}\frac{1}{\ell} + F''_{23} = 0, \quad (20)$$

and

$$\frac{\partial^2 S}{\partial W^2}F'_1 + F''_{11}\frac{\mu_1^2}{\ell^2} + 2F''_{31}\frac{\mu_1}{\ell} + F''_{33} = 0. \quad (21)$$

We introduce the parameter

$$\mu_2 = \ell U\frac{\partial^2 S}{\partial W^2} \quad (22)$$

and, keeping in mind the definition (10), rewrite (21) in the form

$$F''_{11}\frac{\mu_1^2}{\ell^2} + 2F''_{31}\frac{\mu_1}{\ell} + F''_{33} = \frac{\mu_2}{U\tau_0\ell}. \quad (23)$$

It is useful to consider exactly how the two parameters μ_1 and μ_2 are related to the *angular response* of the anemometer. We define the angular response function $g(U, \vartheta)$ for a given wind speed as the ratio of the output $S + s_\vartheta$ when the wind vector $\mathbf{U}$ forms the angle ϑ with the rotor plane. The output S when $\vartheta = 0$ is

$$g(U, \vartheta) = \frac{(S + s_\vartheta)}{S} = 1 + \frac{s_\vartheta}{S}, \quad (24)$$

where s_ϑ is the output increment for a given value of ϑ. In Fig. 2 a cup anemometer is mounted for the determination of the angular response. We consider ϑ positive if the projection w_ϑ of $\mathbf{U}$ on the anemometer axis points away from the bulky part of the anemometer body.

Denoting the projection of $\mathbf{U}$ on the rotor plane $U + u_\vartheta$ and considering only small angles ($\vartheta \ll 1$), we have

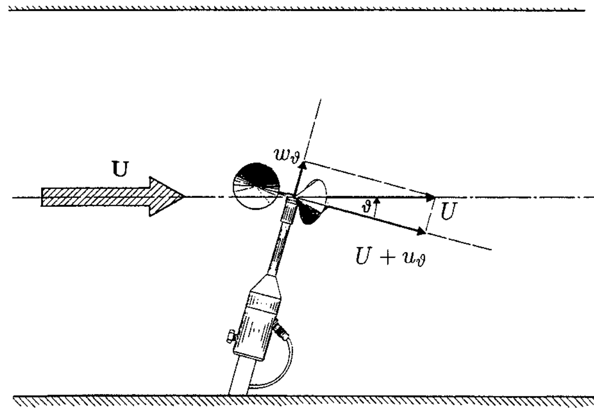

FIG. 2. Cup anemometer mounted in a wind tunnel with the purpose of determining its angular response characteristics. As indicated, ϑ is positive in this case.

$$u_\vartheta = U\cos(\vartheta) - U \approx -U\frac{\vartheta^2}{2} \quad (25)$$

and

$$w_\vartheta = U\sin(\vartheta) \approx U\vartheta. \quad (26)$$

We may determine s_ϑ to the first order in ϑ by substituting (25) and (26) into (13), which yields

$$s_\vartheta = \mu_1\frac{U}{\ell}\vartheta + O(\vartheta^2), \quad (27)$$

where $O(x)$ means of order x.

Similarly, s_ϑ to the second order in ϑ can be determined by means of (18). Again, substituting (25) and (26) and, in addition, the first-order result in (27), we obtain

$$\begin{aligned}\ell s_\vartheta &= \underbrace{u_\vartheta}_{O(\vartheta^2)} + \underbrace{\mu_1 w_\vartheta}_{O(\vartheta)} \\ &\quad + \frac{\ell\tau_0}{2}\{\underbrace{F''_{22}u_\vartheta^2}_{O(\vartheta^4)} + \underbrace{2F''_{23}u_\vartheta w_\vartheta}_{O(\vartheta^3)} + \underbrace{F''_{33}w_\vartheta^2}_{O(\vartheta^2)}\} \\ &\quad + \frac{\ell\tau_0}{2}\{\underbrace{F''_{11}s_\vartheta^2}_{O(\vartheta^2)} + \underbrace{2F''_{12}s_\vartheta u_\vartheta}_{O(\vartheta^3)} + \underbrace{2F''_{31}w_\vartheta s_\vartheta}_{O(\vartheta^2)}\} \\ &\approx \mu_1 U\vartheta - U\frac{\vartheta^2}{2} \\ &\quad + U^2\ell\tau_0\left(F''_{11}\frac{\mu_1^2}{\ell^2} + 2F''_{31}\frac{\mu_1}{\ell} + F''_{33}\right)\frac{\vartheta^2}{2} \\ &= U\left[\mu_1\vartheta - (1 - \mu_2)\frac{\vartheta^2}{2}\right], \end{aligned} \quad (28)$$

where we have used the constraint in (23). According to the definition in (24) we now find

$$g(U, \vartheta) = 1 + \frac{U}{U - U_0}\left[\mu_1\vartheta - (1 - \mu_2)\frac{\vartheta^2}{2}\right]. \quad (29)$$

An ideal cup anemometer responds only to the wind components in the rotor plane, and this means that

$$S + s_\vartheta = Sg(U, \vartheta) = \frac{U\cos(\vartheta) - U_0}{\ell} \quad (30)$$

or, since $S = (U - U_0)/\ell$,

$$U\left[1 + \mu_1\vartheta - (1 - \mu_2)\frac{\vartheta^2}{2}\right] = U\cos(\vartheta). \quad (31)$$

Since $\cos(\vartheta) = 1 - \vartheta^2/2$ to the second order in ϑ, (31) implies that for an ideal cup anemometer $\mu_1 = \mu_2 = 0$. This is not the case for a real cup anemometer, but, based on the experimental determination of the angular response of the Risø cup anemometer (Busch et al. 1980), it seems reasonable to assume that μ_1 and μ_2 are independent of wind speed.

c. *Experimental determinations of the expansion coefficients*

At this point it will be appropriate to introduce the model of Wyngaard et al. (1974). They normalized the perturbation variables as $s' = s/S$, $u' = u/U$, and $w' = w/U$ and formulated the second-order expansion as follows:

$$s' + \tau_0\dot{s}' = a_1u' + a_2w' + a_3s'^2 + a_4u'^2 + a_5w'^2 + a_6s'u' + a_7u'w' + a_8w's', \quad (32)$$

where the term corresponding to lateral perturbations $v' = v/U$ was omitted since it was not relevant in their study. They were able to determine the eight dimensionless coefficients in (32) for a particular cup anemometer by means of a wind tunnel in which they let the rotor have a different angular velocity than that corresponding to the flow velocity and at the same time measure the torque on the rotor. By tilting the cup axis they produced a pseudovertical velocity component and so determined a_2, a_5, a_7, and a_8.

Wyngaard et al. (1974) found $a_1 = 1.03 \pm 10\%$. Assuming that this means that $a_1 = 1$ within the experimental uncertainty, we conclude, by comparing (32) with (18), that $U/(U - U_0) \equiv a_1 = 1$; that is, that we may consider $U_0 = 0$. With this simplification we can identify the two equations, term by term, and write the result of Wyngaard et al. (1974) as

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \mu_1 \\ F''_{11}U\tau_0/(2\ell) \\ F''_{22}U\tau_0\ell/2 \\ F''_{33}U\tau_0\ell/2 \\ F''_{12}U\tau_0 \\ F''_{23}U\tau_0\ell \\ F''_{31}U\tau_0 \end{Bmatrix} = \begin{Bmatrix} 1.03 \pm 10\% \\ 0.06 \pm 0.1 \\ -0.23 \pm 10\% \\ 0.96 \pm 10\% \\ 0.67 \pm 0.1 \\ -0.73 \pm 10\% \\ 0.04 \pm 0.1 \\ -0.12 \pm 0.1 \end{Bmatrix}. \quad (33)$$

TABLE 1. The result of the Coppin (1982) measurements on seven anemometers.

Type	a_1	a_3	a_4	a_5	a_6
Friedrichs	1.27	0.00	1.10	0.85	−1.17
Siggelkow	1.18	−0.05	1.03	0.90	−0.90
Teledyne 3	1.25	−0.19	1.11	0.82	−0.86
Teledyne 6	1.26	−0.20	1.15	0.47	−0.89
Gill 3	1.22	−0.20	1.14	≈0	−0.79
Casella	1.19	−0.29	0.98	≈0	−0.69
Thies	1.18	−0.11	1.18	0.95	−1.26

The left equation, together with the constraints in (19), (20), and (23), and the assumption that $a_1 = 1$, implies the following relations between the coefficients in (32):

$$a_3 + a_4 + a_6 = 0, \quad (34)$$

$$a_2(2a_3 + a_6) + a_7 + a_8 = 0, \quad (35)$$

and

$$a_2^2a_3 + a_2a_8 + a_5 = \mu_2/2. \quad (36)$$

We note that (34) and (35) are fulfilled well within the experimental errors by the right equation of (33). The left-hand sides of those two equations become 0.00 ± 0.12 and -0.15 ± 0.19, respectively.

In all theoretical models it has been assumed that the forcing F on the rotor is a second-order polynomial in its variables, that is, the wind velocity and the rotation rate. It is easily seen that, to the extent we can neglect the starting speed U_0, the time constant given by (10) is inversely proportional to the wind speed U and all the second derivatives of F are independent of U. The first implication is that not only a_1 and a_2 but all the constants $a_1, \ldots, a_8$ are independent of U, the second that the so-called distance constant

$$L = U\tau_0 \quad (37)$$

is a true instrument length constant just as the response distance ℓ. Practical experience with cup anemometers confirms that L is independent of U.

The Coppin (1982) results are shown in Table 1. This investigation is somewhat limited since a_2, a_7, and a_8 were not determined for the seven different cup anemometers. Further, in all cases a_1 seems to be significantly different from unity, implying that the wind speed is so low that $a_1 = U/(U - U_0)$ cannot be considered independent of U which, in turn, means that (34), (35), and (36) should be generalized accordingly. Even then (34) seems reasonably well fulfilled, and we note that in all cases a_3 is negative, just as stated by Wyngaard et al. (1974).

We see that the four instrument constants ℓ, L, μ_1, and μ_2 are important to characterize the response, the (spatial) filtering characteristics, and the angular response of the cup anemometer.

d. Overspeeding

Now we have the tools to quantify the overspeeding. Taking the average of (18) and assuming $U_0 = 0$, we obtain the expression for the relative overspeeding

$$\frac{\langle s\rangle}{S} = \frac{\langle v^2\rangle}{2U^2} + \frac{\ell L}{2}\left(F''_{22}\frac{\langle u^2\rangle}{U^2} + 2F''_{23}\frac{\langle uw\rangle}{U^2} + F''_{33}\frac{\langle w^2\rangle}{U^2}\right) + \frac{L}{2}\left(\frac{F''_{11}}{\ell}\frac{\langle s^2\rangle}{S^2} + 2F''_{12}\frac{\langle su\rangle}{SU} + 2F''_{31}\frac{\langle ws\rangle}{SU}\right), \tag{38}$$

where we have used (37). To evaluate the right-hand side we must know not only the variances $\langle u^2\rangle$, $\langle v^2\rangle$, $\langle w^2\rangle$, and the covariance $\langle uw\rangle$ of the turbulent wind but also the variance $\langle s^2\rangle$ of the anemometer response and the covariances $\langle su\rangle$ and $\langle ws\rangle$. We have to apply a bootstrap operation to resolve this dilemma and use the solution to the first-order equation (13) with the initial condition $s(-\infty) = 0$:

$$s(t) = \frac{1}{\ell}\int_0^\infty [u(t-\tau) + \mu_1 w(t-\tau)]e^{-\tau/\tau_0}\frac{d\tau}{\tau_0}. \tag{39}$$

Now we get

$$\langle su\rangle = \frac{1}{\ell}\int_0^\infty [R_u(\tau) + \mu_1 R_{wu}(\tau)]e^{-\tau/\tau_0}\frac{d\tau}{\tau_0} \tag{40}$$

and

$$\langle ws\rangle = \frac{1}{\ell}\int_0^\infty [R_{uw}(\tau) + \mu_1 R_w(\tau)]e^{-\tau/\tau_0}\frac{d\tau}{\tau_0}, \tag{41}$$

where we have introduced the covariance functions $R_u(\tau) \equiv \langle u(t)u(t+\tau)\rangle$, $R_w(\tau) \equiv \langle w(t)w(t+\tau)\rangle$, and $R_{uw}(\tau) \equiv \langle u(t)w(t+\tau)\rangle$.

At this point it seems more practical to rewrite the last two equations in terms of the wavenumber spectra, $F_u(k)$ and $F_w(k)$, and the cross spectrum $F_{uw}(k) \equiv \mathrm{Co}(k) + iQ(k)$, where $\mathrm{Co}(k)$ and $Q(k)$ are the co- and quadrature-spectra, respectively, employing Taylor's "frozen turbulence" hypothesis to convert integrals over frequency to integrals over the wavenumber component k in the flow direction. The definition of a spectrum is

$$F_\beta(k) = \frac{U}{2\pi}\int_{-\infty}^\infty R_\beta(\tau)\exp(-ikU\tau)\,d\tau, \tag{42}$$

where $\beta = u$, v, w, or uw.

Once $\langle su\rangle$ and $\langle ws\rangle$ are determined, $\langle s^2\rangle$ is easily obtained by multiplying (13) by s and averaging, keeping in mind that $2\langle s\dot{s}\rangle = d\langle s^2\rangle/dt = 0$:

$$\langle s^2\rangle = (\langle su\rangle + \mu_1\langle ws\rangle)/\ell. \tag{43}$$

Rewriting (40) and (41) in terms of the spectra and, subsequently, applying (43), we obtain

$$\langle \mathrm{su}\rangle = \frac{1}{\ell}\int_{-\infty}^\infty \frac{F_u(k)\,dk}{1 + k^2L^2} + \frac{\mu_1}{\ell}\int_{-\infty}^\infty \frac{\mathrm{Co}(k) + kLQ(k)}{1 + k^2L^2}\,dk, \tag{44}$$

$$\langle \mathrm{ws}\rangle = \frac{\mu_1}{\ell}\int_{-\infty}^\infty \frac{F_w(k)\,dk}{1 + k^2L^2} + \frac{1}{\ell}\int_{-\infty}^\infty \frac{\mathrm{Co}(k) - kLQ(k)}{1 + k^2L^2}\,dk, \tag{45}$$

and

$$\langle s^2\rangle = \frac{1}{\ell^2}\int_{-\infty}^\infty \frac{F_u(k)\,dk}{1 + k^2L^2} + \frac{\mu_1^2}{\ell^2}\int_{-\infty}^\infty \frac{F_w(k)\,dk}{1 + k^2L^2} + \frac{\mu_1}{\ell^2}\int_{-\infty}^\infty \frac{2\mathrm{Co}(k)\,dk}{1 + k^2L^2}. \tag{46}$$

Inserting in (38), using the left equation of (33) and combining with (34), (35), and (36), we obtain, after some manipulation,

$$\frac{\langle s\rangle}{S} = \underbrace{\frac{\langle v^2\rangle}{2U^2}}_{\mathrm{A}} + \underbrace{\frac{a_4}{U^2}\int_{-\infty}^\infty \frac{k^2L^2F_u(k)}{1 + k^2L^2}\,dk}_{\mathrm{B}} + \underbrace{\frac{a_5}{U^2}\int_{-\infty}^\infty \frac{k^2L^2F_w(k)}{1 + k^2L^2}\,dk}_{\mathrm{C}} + \underbrace{\frac{\mu_2}{2U^2}\int_{-\infty}^\infty \frac{F_w(k)\,dk}{1 + k^2L^2}}_{\mathrm{D}} + \underbrace{\frac{a_7}{U^2}\int_{-\infty}^\infty \frac{k^2L^2\mathrm{Co}(k)}{1 + k^2L^2}\,dk}_{\mathrm{E}} + \underbrace{\frac{a_2a_6 - a_8}{U^2}\int_{-\infty}^\infty \frac{kLQ(k)}{1 + k^2L^2}\,dk}_{\mathrm{F}}. \tag{47}$$

This quite general result was obtained by Kristensen (1993). However, the second term (B), which has always been considered the most intriguing, was obtained already by Kaganov and Yaglom (1976) and Busch and Kristensen (1976). This term is what is usually considered *overspeeding* but, as we shall see, it is seldom the most important bias. The first term (A) is the contribution to the bias caused by the wind direction fluctuations. The terms C and D are both proportional to the variance of the vertical velocity component $\tilde{w}$. We see that term C is essentially proportional to the variance from wavenumbers larger than the reciprocal of the distance constant L, whereas term D is the contribution from $\tilde{w}$ variance at wavenumbers smaller than $1/L$. Equations (33) and (36) indicate that $\mu_2 \simeq 2a_5$ so that, at least approximately, the sum of terms C and D becomes $a_5\langle w^2\rangle/(2U^2)$. Terms E and F are proportional to the covariance of $\tilde{u}$ and $\tilde{w}$ at high wavenumbers. Term E is the "in-phase" and term F the "out-of-phase" contribution. Both terms are probably small, judging from (33).

3. Phenomenological forcing model

Instead of stating the result (47) by means of the parameters $a_1, \ldots, a_8$, we may wish to formulate this result in terms of ''physical'' parameters characterizing the cup anemometer. This can be accomplished by postulating a model of the forcing on the cup rotor. We will assume that the friction in the bearings can be neglected with the implication that U_0 can be considered zero and that the entire forcing is due to the wind. Further, the model should take into account what we know of the instrument response and the first-order response characteristics to a change wind speed.

Leaving out for the moment the vertical velocity component and introducing for convenience the notation

$$\tilde{h} = (\tilde{u}^2 + \tilde{v}^2)^{1/2} \tag{48}$$

for the instantaneous, total horizontal wind speed component, we hypothesize, along with many others (see, e.g., Kaganov and Yaglom 1976), a form of the function $F(\tilde{s}, \tilde{h}, 0)$ based on the assumption that the torque of the turbulent wind is a homogeneous second-order polynomial in $\tilde{s}$ and $\tilde{h}$. We add the constraint that one of the two roots in $F(S, U, 0) = 0$ is $S = U/\ell$ [(5) with $U_0 = 0$] and exclude the possibility that $S = U/\ell$ is a double root because this would imply that $F_1'(S, U, 0)$ is zero, corresponding to an infinite timescale τ_0 according to (10). All this suggests the following form:

$$F(\tilde{s}, \tilde{h}, 0) = \frac{1}{2}\rho C \frac{Ar}{I}(\tilde{h} - \ell\tilde{s})(\tilde{h} + \Lambda\tilde{s}). \tag{49}$$

Here, C is a dimensionless constant; ρ, the density of air; A, an effective cup area; r, an effective radius of the cup rotor; and I, its moment of inertia. The quantity Λ is some instrument length scale. If it is positive, there is only one positive root in $F(S, U, 0) = 0$, namely, that corresponding to the response. Later we will argue, on basis of the measurements by Wyngaard et al. (1974) and Coppin (1982), that Λ is indeed positive.

The first useful result we can derive from (49) is an expression for the time constant τ_0, which we introduced with (10). Taking the derivative with respect to $\tilde{s}$, we get

$$\tau_0 = \frac{-1}{F_1'} = \frac{1}{U}\frac{2I}{\rho CAr(\ell + \Lambda)}. \tag{50}$$

We note that the time constant is inversely proportional to U with a factor of proportionality that is determined entirely by the density of air and of the rotor material and by the rotor geometry. Therefore, we get the expression

$$L = U\tau_0 = \frac{2I}{\rho CAr(\ell + \Lambda)} \tag{51}$$

for the distance constant that, since it does not depend on the wind speed, is a true instrument constant. The fact that a cup anemometer response is characterized by a distance constant rather than a time constant has been confirmed experimentally (MacCready 1965).

Since the moment of inertia I is proportional to the rotor density ρ_r and to the fifth power of its linear dimensions ($\propto r$), we conclude from (51) that the distance constant is proportional to ρ_r/ρ and $r^2/(\ell + \Lambda)$. The response distance ℓ is independent of the moment of inertia—that is, of the density of the rotor material—and determined entirely by the rotor geometry. It is proportional to r and usually of the same order. If we assume that Λ scales with r, just like ℓ does, we conclude that the distance constant is proportional to r. The rotor of the Risø-70 model (Busch et al. 1980) is made of carbon reinforced plastic with a density ρ_r of about 1.5 g cm^{-3}. The radius r is 0.07 m and the distance constant L was determined to be 1.7 m. This is a rather typical modern, sturdy cup anemometer, which is used for routine measurements of mean wind speed by Risø National Laboratory. Older models are typically made of steel and they are often larger, with radii of about 0.15 m. They have distance constants of about 20 m and consequently react much slower than newer models of standard cup anemometers. The smallest distance constant is reported by Frenzen (1988). His miniature instrument, which is designed for turbulence measurements and is quite fragile and unsuited for routine operations, has $L = 0.25$ m. The rotor is made of styrofoam and has a radius of only 0.01 m.

Applying (51), the forcing function may now be rewritten as

$$F(\tilde{s}, \tilde{h}, 0) = \frac{(\tilde{h} - \ell\tilde{s})(\tilde{h} + \Lambda\tilde{s})}{L(\ell + \Lambda)}. \tag{52}$$

We see that the forcing by the *horizontal* wind can be specified by means of the three length scales: ℓ, L, and Λ. From the left equation of (33), we get

$$\begin{Bmatrix} a_3 \\ a_4 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} \dfrac{F''_{11}}{2}\dfrac{L}{\ell} \\ \dfrac{F''_{22}}{2}\ell L \\ F''_{12}L \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\Lambda}{\ell + \Lambda} \\ \dfrac{\ell}{\ell + \Lambda} \\ -\dfrac{\ell - \Lambda}{\ell + \Lambda} \end{Bmatrix}. \tag{53}$$

The data in (33) by Wyngaard et al. (1974) and those by Coppin (1982) show that in all the cases a_3 is never positive, a_4 is always positive, and a_6 is always negative. Assuming this to be the case in general, (53) implies that $\ell + \Lambda > 0$ (since $a_4 > 0$ and $\ell > 0$) and, consequently, that $\Lambda \geq 0$ (since $a_3 < 0$). Further, since $a_6 < 0$, we conclude that $\ell > \Lambda$ and that

$$a_4 < 1. \tag{54}$$

This inequality is, according to (33), fulfilled by the data by Wyngaard et al. (1974). Coppin (1982), on the other hand, found that six out of the seven anemometers he investigated had values of a_4 larger than 1. At the same time he found that a_1 was larger than 1 by about 0.2 for all seven anemometers. Our model predicts that a_1 is exactly 1 and that the only possible way to understand the results by Coppin (1982) in our framework is to accept that the offset speed U_0 is significant. It is possible to take U_0 into account in our model, and one consequence of this generalization is that a_4 must be divided by a_1^2 before comparison with the inequality in (54). When this ''normalization'' is carried out, all his values of a_4 fall well below 1.

We see that the model fulfills (34). However, our model predicts still another constraint:

$$a_4 - a_3 = 1. \tag{55}$$

The data given by Wyngaard et al. (1974) show that $a_4 - a_3 = 1.19 \pm 14\%$ and those from Coppin (1982) show corresponding values between 1.08 and 1.29.

Inserting (53) in (20) and (23), these relations take the forms

$$\frac{1}{\ell}F''_{31} + F''_{23} = \frac{\mu_1}{\ell L} \tag{56}$$

and

$$2\frac{\mu_1}{\ell}F''_{31} + F''_{33} = \frac{\mu_2}{\ell L} + \frac{\mu_1^2}{\ell L}\frac{2\Lambda}{\ell + \Lambda}. \tag{57}$$

It is tempting to formulate a model for the torque that includes the dependence on the vertical velocity component $\tilde{w}$. We will do it by securing that the constraints in (56) and (57) are fulfilled.

We note, by looking back at, for example, (13), that a small vertical wind component W has the same effect on the rotation rate as the increase $\mu_1 W$ in the horizontal wind component. The first thing to do would therefore be to replace $\tilde{h}$ in (52) by $\tilde{h} + \mu_1\tilde{w}$. This is sufficient to fulfill (56), but not (57). To accomplish that, without violating (56) or any other constraints we have used, all we have to do is to add a term proportional to $\tilde{w}^2$. It is easily seen that the expression

$$F(\tilde{s}, \tilde{h}, \tilde{w}) = \frac{(\tilde{h} + \mu_1\tilde{w} - \ell\tilde{s})(\tilde{h} + \mu_1\tilde{w} + \Lambda\tilde{s})}{L(\ell + \Lambda)} + \frac{\mu_2\tilde{w}^2}{2L\ell} \tag{58}$$

meets all our requirements.

With this equation for the torque we can now establish the complete translation table between the coefficients $a_1, \ldots, a_8$ and the model parameters. Carrying out all the differentiations of $F(\tilde{s}, \tilde{h}, \tilde{w})$ to second order and then setting $(\tilde{s}, \tilde{h}, \tilde{w}) = (U/\ell, U, 0)$, we get the result

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \equiv \begin{Bmatrix} 1 \\ \mu_1 \\ \dfrac{F''_{11}}{2}\dfrac{L}{\ell} \\ \dfrac{F''_{22}}{2}\ell L \\ \dfrac{F''_{33}}{2}\ell L \\ F''_{12}L \\ F''_{23}\ell L \\ F''_{31}L \end{Bmatrix} = \begin{Bmatrix} 1 \\ \mu_1 \\ -\dfrac{\Lambda}{\ell + \Lambda} \\ \dfrac{\ell}{\ell + \Lambda} \\ \dfrac{\mu_1^2\ell}{\ell + \Lambda} + \dfrac{\mu_2}{2} \\ -\dfrac{\ell - \Lambda}{\ell + \Lambda} \\ \dfrac{2\mu_1\ell}{\ell + \Lambda} \\ -\mu_1\dfrac{\ell - \Lambda}{\ell + \Lambda} \end{Bmatrix}. \tag{59}$$

This phenomenological model shows that the dynamic behavior of a cup anemometer can be described to second order in perturbations around a response point $(S, U, W) = (U/\ell, U, 0)$ by five independent parameters. The four traditional parameters are specified as the response distance ℓ, which specifies the translation from signal to wind speed; the distance constant L, which characterizes the low-pass filtering of the anemometer; and the two dimensionless parameters μ_1 and μ_2, which account for the sensitivity to the vertical wind component. They can be obtained by use of a low-turbulence wind tunnel. This is obvious where ℓ, μ_1, and μ_2 are concerned. The distance constant L can be determined by holding and releasing the cup rotor in a constant wind. The fifth parameter, the length scale Λ, is difficult to interpret. It does not manifest itself in the steady-state response or in the first-order dynamics as is the case with the other four parameters. The model also predicts the following constraints:

$$a_2a_6 - a_8 = 0, \tag{60}$$

and

$$2a_2a_4 - a_7 = 0, \tag{61}$$

which are both consistent with the results (33) by Wyngaard et al. (1974).

a. Reformulating the overspeeding

We may now formulate the general result (47) by means of (59). We note that, by virtue of (60), the last term containing the quadrature spectrum disappears, and we obtain for the relative overspeeding $\delta \equiv \langle s \rangle$:

$$\delta = \underbrace{\frac{\ell}{\ell + \Lambda}\frac{1}{U^2}\int_{-\infty}^{\infty}\frac{k^2L^2F_u(k)}{1 + k^2L^2}\,dk}_{\delta_u} + \underbrace{\frac{\langle v^2\rangle}{2U^2}}_{\delta_v}$$
$$+ \underbrace{\mu_2\frac{\langle w^2\rangle}{2U^2} + \frac{\mu_1^2\ell}{\ell + \Lambda}\frac{1}{U^2}\int_{-\infty}^{\infty}\frac{k^2L^2F_w(k)}{1 + k^2L^2}\,dk}_{\delta_w}$$
$$+ \underbrace{\frac{2\mu_1\ell}{\ell + \Lambda}\frac{1}{U^2}\int_{-\infty}^{\infty}\frac{k^2L^2\mathrm{Co}(k)}{1 + k^2L^2}\,dk}_{\delta_*}, \tag{62}$$

where we have written the entire bias δ, that is, overspeeding, in terms of four distinct types of biases, namely, three related to each of the velocity components and one to the covariance between the velocity components u and w.

We see that the u bias, δ_u, is always positive and proportional to the variance of u after a first-order high-pass line filtering along the direction of the mean wind speed with a length constant equal to the distance constant. There is a straightforward interpretation of this result: The cup anemometer is fast enough to resolve correctly the contribution to the mean wind from eddies larger than the distance constant L; the smaller eddies will give rise to a positive bias.

In contrast to δ_u, the v bias, δ_v, gets contributions from the v variance at all wavenumbers.

The w bias, δ_w, can be written as a sum of two terms: the first proportional to the entire w variance and the second, which is never negative, to the high wavenumber part of the w variance. We see that the first term, which has the same sign as μ_2, is similar to the v bias and the second to the u bias.

Finally δ_*, which we will term the stress bias, is proportional to the high wavenumber part of the cospectrum of u and w.

b. *Quantifying the overspeeding*

If L is small compared to the scale of the turbulence, and this means essentially small compared to the height z of the measurement, we may assume that the turbulence is locally isotropic and, according to many researchers, for example, Kaimal et al. (1972) and Wyngaard and Coté (1972), given by

$$F_u(k) = \frac{\alpha_1}{2}\varepsilon^{2/3}|k|^{-5/3}, \tag{63}$$

$$F_w(k) = \frac{2\alpha_1}{3}\varepsilon^{2/3}|k|^{-5/3}, \tag{64}$$

and

$$\mathrm{Co}(k) = -\frac{\zeta_1}{2}\frac{dU}{dz}\varepsilon^{1/3}|k|^{-7/3}. \tag{65}$$

Here, $\alpha_1 \approx 0.56$ (Kristensen et al. 1989) is the Kolmogorov constant, and $\zeta_1 \approx 0.15$ is a corresponding dimensionless constant for the cospectrum. The rate of dissipation of specific kinetic energy ϵ enters in both spectra. In addition, the cospectrum contains the wind shear dU/dz as a factor.

The integrals in (62) can therefore easily be evaluated, but, inspired by (33) and by experience with the Risø anemometer (Busch et al. 1980), we assume that $\mu_1 = a_2 \approx 0$, so that the only integrals to be evaluated are those pertaining to δ_u and δ_*. We get

$$\delta_u = \frac{\pi}{\sqrt{3}}\alpha_1\frac{\ell}{\ell + \Lambda}\frac{(\varepsilon L)^{2/3}}{U^2} \tag{66}$$

$$\delta_* = -\frac{\pi}{\sqrt{3}}\frac{2\mu_1\ell}{\ell + \Lambda}\zeta_1\frac{1}{U^2}\frac{dU}{dz}L(\varepsilon L)^{1/3}. \tag{67}$$

If the atmospheric surface layer is neutrally stratified, it is quite easy to formulate the four biases in terms of the surface friction velocity u_* and z. Since $dU/dz = u_*/(\kappa z)$, and $\epsilon = u_*^3/(\kappa z)$, $\kappa \approx 0.4$ being the von Kármán constant, and since, according to the summary by Panofsky and Dutton (1984), $\langle u^2\rangle/u_*^2 \approx 5.7$, $\langle v^2\rangle/u_*^2 \approx 3.7$ and $\langle w^2\rangle/u_*^2 \approx 1.7$, we get

$$\delta_u = a_4\frac{\pi}{\sqrt{3}}\alpha_1\frac{u_*^2}{U^2}\left(\frac{L}{\kappa z}\right)^{2/3} \approx 2a_4\left(\frac{L}{z}\right)^{2/3}\frac{u_*^2}{U^2}, \tag{68}$$

$$\delta_v = \frac{\langle v^2\rangle}{2U^2} \approx 2\frac{u_*^2}{U^2}, \tag{69}$$

$$\delta_w = \mu_2\frac{\langle w^2\rangle}{2U^2} \approx 1.5a_5\frac{u_*^2}{U^2}, \tag{70}$$

and

$$\delta_* = -a_7\frac{\pi}{\sqrt{3}}\zeta_1\frac{u_*^2}{U^2}\left(\frac{L}{\kappa z}\right)^{4/3} \approx -a_7\left(\frac{L}{z}\right)^{4/3}\frac{u_*^2}{U^2}, \tag{71}$$

where we have used the translating table (59) to express—for convenience—the result in terms of the a coefficients by Wyngaard et al. (1974).

As stated by Frenzen (1988), a cup anemometer can be constructed to have ideal angular response, that is, $\mu_1 \approx \mu_2 \approx 0$ for a wind inclination within $\pm 20°$. In that case, (59) shows that $a_5 \approx 0$ and $a_7 \approx 0$. This means that δ_w and δ_* can be neglected for such an anemometer. The coefficient a_4 of the u bias must always be positive; otherwise, the cup rotor will not start, as can easily be seen. However, by making the distance constant L small enough, δ_u can be reduced significantly. For example, if we assume that $a_4 \approx 1$, $L = 2$ m, $z_0 = 0.05$ m, and $z = 10$ m, then $\delta_u \approx 0.4\%$, whereas $\delta_v \approx 1\%$.

The situation changes dramatically if we go from a neutral to a strongly unstable stratification. Kristensen (1993) determined the four biases in a horizontally homogeneous, unstably stratified atmospheric surface and showed that δ_v could conceivably be as large as 18% at $z = 5.66$ m in the Kansas experiment (Kaimal and Wyngaard 1990), whereas the other biases probably did not exceed about 2% (δ_* about -0.01%) under the same circumstances. This is intimately connected to the facts that δ_u and δ_* contain a weighting factor $(L/z)^\nu$, where ν is 2/3 in the first case and 4/3 in the second, and that the variance of w is length-scale limited in contrast to the variance of v (and u).

c. Suppressing the overspeeding

In the preceding section it was shown

1) that δ_u is usually small, and that the key parameter to its further reduction is the factor $(L/z)^{2/3}$;
2) that δ_w is limited because the variance of w is limited by the turbulence scale that, in turn, is proportional to the height z, and that this bias can be reduced by designing the cup anemometer such that μ_2 is close to zero; and
3) that δ_* is generally small mainly because of the factor $(L/z)^{4/3}$, but also because for most cup anemometers μ_1 is small.

However, *the v bias cannot be reduced by an appropriate anemometer design.* In a way δ_v is not a bias at all if we were only considering the mean of the wind *speed* and not the magnitude of the wind *velocity vector.* The last interpretation is prevailing and, consequently, we must look into the possibility of reducing δ_v. Fortunately, it seems that this can be accomplished by a data processing that, by use of a wind vane, takes into account the actual fluctuations of the wind direction.

If there were no wind direction fluctuations, we could determine the mean wind speed over the period T by counting the number N of revolutions. Then, assuming that the offset speed U_0 negligible, the mean wind speed would be given by $U = 2\pi\ell \times N/T$ since N air columns of length $2\pi\ell$ has run through the anemometer in the time T. If there are wind direction fluctuations, we record, instead of "counting one," the instantaneous angle of direction $\tilde{\phi}$ every time the rotor has turned one full revolution. Then we can, after the elapse of the time T, obtain the mean vector wind by

$$\begin{Bmatrix} U_E \\ U_N \end{Bmatrix} = \frac{2\pi\ell}{T}\begin{Bmatrix} \sum_{i=1}^{N} \cos(\tilde{\phi}_i) \\ \sum_{i=1}^{N} \sin(\tilde{\phi}_i) \end{Bmatrix}. \qquad (72)$$

The procedure is illustrated graphically in Fig. 3. The mean wind velocity and direction are now given by

$$U = \sqrt{U_E^2 + U_N^2} \qquad (73)$$

and

$$\langle\tilde{\phi}\rangle = \arctan\left(\frac{U_N}{U_E}\right). \qquad (74)$$

This procedure reduces δ_v significantly since is corresponds to a high-pass filtering of the fluctuating, lateral velocity component. The "apparent" variance of v that enters in the expression for δ_v is given by

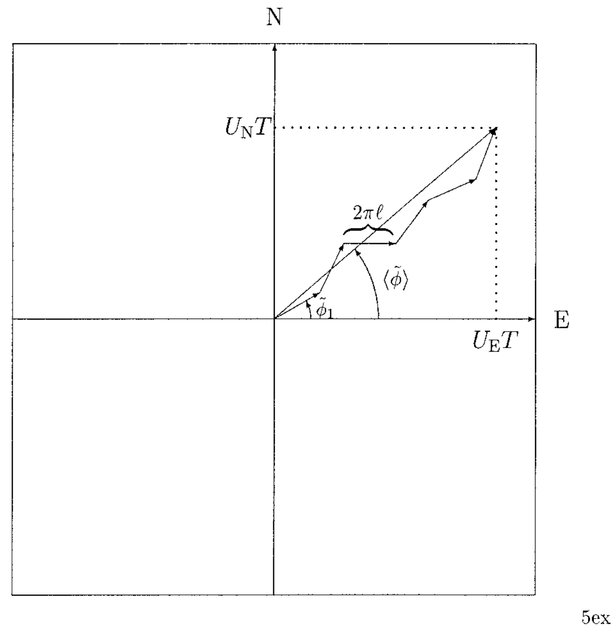

FIG. 3. Illustration of the vector wind-run method. The orientation is such that the direction is measured counterclockwise from the east.

$$\langle v^2\rangle_{\text{app}} = 2\int_{-\infty}^{\infty}\left\{1 - \frac{\sin(2\pi k\ell)}{2\pi k\ell}\right\}F_v(k)\,dk, \qquad (75)$$

where $F_v(k)$ is the spectrum of the lateral velocity component. Using this signal processing, we have reduced what MacCready has called the DP-error (for data processing error) (MacCready 1966). We note that the characteristic length constant in (75) is the response distance ℓ. Since this quantity is less than the distance constant L, we can, to an even better approximation than that used in the derivation of (66) and (67), assume that $F_v(k)$ is given by the inertial subrange expression. This is the same as that for $F_w(k)$ in (64), and the v bias becomes

$$\begin{aligned}\delta_v &= \frac{\langle v^2\rangle_{\text{app}}}{2U^2} \\ &= \frac{8}{3}\alpha_1\frac{(2\pi\varepsilon\ell)^{2/3}}{2U^2}\int_0^\infty \{\xi - \sin\xi\}\xi^{-8/3}\,d\xi; \\ &= \frac{3}{5}(2\pi)^{2/3}\Gamma\left(\frac{1}{3}\right)\alpha_1\frac{(\varepsilon\ell)^{2/3}}{U^2} \approx 3.1\frac{(\varepsilon\ell)^{2/3}}{U^2} \\ &\approx 5.6\left(\frac{\ell}{z}\right)^{2/3}\frac{u_*^2}{U^2}. \end{aligned} \qquad (76)$$

We conclude that δ_v now becomes smaller than even δ_u if $\ell < L \ll z$. Figure 4 shows graphically the relative importance of the various biases, δ_u, δ_v, δ_w, and δ_*, together with δ_v after the signal processing introduced by (72). Kristensen (1993) pointed out that, as a bonus,

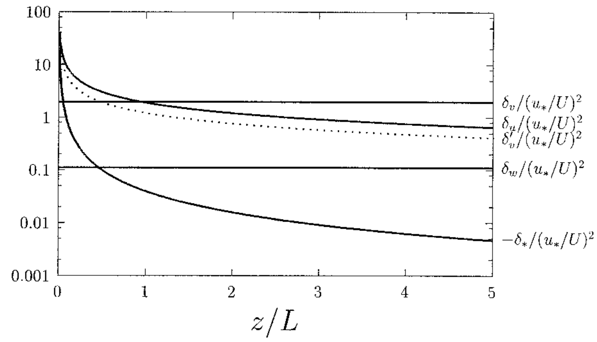

FIG. 4. The relative biases given by (68) through (71) (solid lines) and (76) (dotted line), normalized by $(u_*/U)^2$, as functions of the ratio of the height z and the distance constant L. The actual values of a_4, a_5, and a_7 are taken from Wyngaard et al. (1974) [see (33)], and we have assumed the typical values of the response distance $\ell = 0.2$ m and the distance constant $L = 2$ m.

also the wind direction variance can be obtained using this procedure. By expanding $\langle\cos(\phi)\rangle^2$ and $\langle\sin(\phi)\rangle^2$ in the deviation $\delta\phi$ from $\langle\phi\rangle$, a simple analysis shows

$$\langle\delta\phi^2\rangle = 1 - \langle\cos(\phi)\rangle^2 - \langle\sin(\phi)\rangle^2 + O(\langle\delta\phi^2\rangle^2). \quad (77)$$

4. Conclusions

The cup anemometer has been used for more than 100 yr for routine measurements of the wind velocity. It has turned out to be a very reliable and sturdy instrument, easy to install and operate, with a wind-tunnel calibration that is linear, usually within ± 0.05 m s^{-1}. Further, since it is axisymmetric, it need only be aligned once, namely, at the installation with its axis of rotation vertical.

Using the wind-tunnel calibration in turbulent environment, the mean wind velocity will usually be overestimated. This bias will increase with increasing variance of the three velocity components: $\tilde{u}$ in the mean wind direction, $\tilde{v}$ in the lateral, and $\tilde{w}$ in the vertical direction. In addition, there is a bias related to the covariance between $\tilde{u}$ and $\tilde{w}$ that, depending on the first-order angular response, can be either positive or negative. This bias δ_*, which is proportional to the ratio of the cup anemometer distance constant and the measuring height to the power 4/3, is usually extremely small compared to the other three biases: the u bias, the v bias, and the w bias.

Much effort has been invested in quantifying overspeeding. In an analysis of the Kansas experiment (Kaimal and Wyngaard 1990), it was decided, on basis of an investigation by Izumi and Barad (1970), that the overspeeding was a flat 10%, whereas Kondo et al. (1971) in their analysis concluded that it was quite small; in the worst case, it was less than a few percent. It was very important for the wind-profile measurements in Kansas to measure the wind velocity with an accuracy of about 1%. This was the situation when Kaganov and Yaglom (1976) and, independently, Busch and Kristensen (1976) were able to, on the basis of the empirical model and data by Wyngaard et al. (1974), derive a theoretical expression for the u bias as a function of distance constant L and the scale and the variance of $\tilde{u}$. Kaganov and Yaglom (1976) also derived an expression for the w bias.

Later, the discussion seemed to lose its momentum until the wind-energy community stated stiff requirements concerning the accuracy of the determination of the mean wind speed. As the horizontal flux of wind energy is approximately proportional to the cube of the mean wind speed, an error in this quantity might cause an error three times as large in estimates of wind-energy power. With this in mind, it is easy to understand why it is important to measure the mean wind speed to within a fraction of a percent when the power curve of a wind turbine has to be specified to the accuracy assumed in the circles of manufacturers, consumers, and lawyers in the wind-energy business.

Here, we have analyzed a general, phenomenological model, equivalent to that by Wyngaard et al. (1974), apart from the influence of the lateral velocity fluctuations. We have shown how the linear response leads to certain constraints on the coefficients in a first- and second-order perturbation analysis of the phenomenological equation of motion and to a general equation for overspeeding. An alternative phenomenological model, formulated in terms of the angular response parameters, the response distance, and the distance constant, yielded further constraints, reasonably consistent with the measurements of Wyngaard et al. (1974) and Coppin (1982), and resulted in a less general, but simpler, expression for the overspeeding. On basis of this model, we conclude that cup anemometers can be designed such that the u bias and the w bias are always less than about 1%. However, it takes special signal processing, involving a wind vane, to reduce the v bias or DP error to this level.

It must be emphasized that the overspeeding phenomenon is only relevant when we discuss the mean wind speed. It is possible to use a cup anemometer for measuring the fluctuating, streamwise velocity component with a spatial resolution that corresponds to almost that of a sonic anemometer. Frenzen (1988) has designed a cup anemometer with a distance constant of about 0.25 m and with an almost ideal angular response. For turbulence measurements this instrument can be considered a first-order line filter obeying the equation

$$\frac{ds}{dx} + \frac{s}{L} = \frac{u}{\ell L}. \quad (78)$$

Finally, it should be pointed out that the phenomenological model postulated here might be modified to describe the motion of a propeller-vane anemometer. In this case the anemometer length scale Λ should be set equal to infinity. Kristensen (1994) quantified the overspeeding and found that this anemometer has no u bias, that the v bias and the w bias are the same as those of the cup anemometer, and that there are two more biases to consider. First, the vane is always lagging behind the instantaneous wind direction so that this is never perpendicular to the propeller plane of rotation. Second, the vane is causing the propeller to move with respect to the air.

APPENDIX

List of Symbols

A Effective cup area
$a_1, \ldots, a_8$ Perturbation equation coefficients
C Dimensionless constant of order one
$\mathrm{Co}(k)$ Cospectrum of u and w
f Anemometer factor
F Total forcing on the cup rotor
F'_i Partical derivative of function F with respect to its ith argument
F''_{ij} Partial derivative of function F with respect to its ith and jth arguments
$F_u(k)$ One-dimensional spectrum of u
$F_w(k)$ One-dimensional spectrum of w
$g(U, \vartheta)$ Angular response function
$\tilde{h}$ Instantaneous, horizontal wind velocity component
I Moment of inertia of cup rotor
ℓ Response distance
k Wavenumber component along the x axis
L Distance constant
N Number of cup rotor revolutions during time T
$Q(k)$ Quadrature spectrum of u and w
r Radius of cup rotor
$R_u(\tau)$ Autovariance function of u
$R_w(\tau)$ Autovariance function of w
$R_{uw}(\tau)$ Autovariance function of u and w
$\tilde{s}$ Instantaneous anemometer response
S Anemometer response to constant wind speed U
s $\tilde{s} - S$
s' s/S
$\dot{\tilde{s}}$ Time derivative of $\tilde{s}$ with respect to time t
$\dot{s}$ Time derivative of s with respect to time t
s_ϑ Anemometer response increment for tilt angle ϑ
t Running time
T Averaging time
$(\tilde{u}, \tilde{v}, \tilde{w})$ Instantaneous wind velocity vector in (x, y, z) coordinate system
U Horizontal mean wind velocity component along the x axis
U_0 Offset speed
W Vertical mean wind velocity component along the z axis
U_E, U_N East and north component of mean wind velocity, respectively
(u, v, w) $(\tilde{u} - U, \tilde{v}, \tilde{w})$
(u', v', w') $(u/U, v/U, w/U)$
$\langle v^2 \rangle_{\mathrm{app}}$ Apparent variance of v
u_ϑ Increment of velocity component in the plane of the cup rotor for tilt angle ϑ
w_ϑ Velocity component perpendicular to the plane of the cup rotor for tilt angle ϑ
u_* Surface friction velocity
x, y, z Ground-based Cartesian coordinate system with the x axis in the direction of the mean wind and the z axis vertical
α_1 Kolmogorov constant
δ Total relative overspeeding
δ_u u bias
δ_v v bias
δ_w w bias
δ_* Stress bias
ε Rate of dissipation
ζ_1 Dimensionless constant in the uw cospectrum
ϑ Tilt angle
κ von Kármán constant
Λ Characteristic instrument length scale
μ_1 First-order angular response parameter
μ_2 Second-order angular response parameter
ρ Density of air
ρ_τ Density of cup rotor
τ Time lag
τ_0 Cup anemometer time constant
$\tilde{\phi}$ Instantaneous wind direction in (E, N) coordinates
$\langle \delta\phi^2 \rangle$ Wind direction variance

REFERENCES

Brazier, C. E., 1914: Recherches expérimentales sur les moulenets anémometrique. *Ann. Bur. Centr. Météor. France,* 157–300.

Busch, N. E., and L. Kristensen, 1976: Cup anemometer overspeeding. *J. Appl. Meteor.,* **15,** 1328–1332.

——, O. Christensen, L. Kristensen, L. Lading, and S. E. Larsen, 1980: Cups, vanes, propellers and laser anemometers. *Air-Sea Interaction—Instruments and Methods,* F. Dobson, L. Hasse, and R. Davis, Eds., Plenum Press, 11–46.

Coppin, P. A., 1982: An examination of cup anemometer overspeeding. *Meteor. Rdsch.,* **35,** 1–11.

Frenzen, P., 1988: Fast response cup anemometers for atmospheric turbulence research. *Proc. Eighth Symp. on Turbulence and Diffusion,* San Diego, CA, Amer. Meteor. Soc., 112–115.

Izumi, I., and M. L. Barad, 1970: Wind speeds as measured by cup and sonic anemometers and influenced by tower structure. *J. Appl. Meteor.,* **9,** 851–856.

Kaganov, E. I., and A. M. Yaglom, 1976: Errors in wind speed measurements by rotation anemometers. *Bound.-Layer Meteor.,* **10,** 15–34.

Kaimal, J. C., and J. C. Wyngaard, 1990: The Kansas and Minnesota experiments. *Bound.-Layer Meteor.,* **50,** 31–47.

——, ——, Y. Izumi, and O. R. Coté, 1972: Spectral characteristics of surface layer turbulence. *Quart. J. Roy. Meteor. Soc.,* **98,** 563–589.

Kondo, J., G. Naito, and Y. Fujinawa 1971: Response of cup anemometer in turbulence. *J. Meteor. Soc. Japan,* **49,** 63–74.

Kristensen, L., 1993: The cup anemometer and other exciting instruments. Risø National Laboratory Risø-R-615(EN), Roskilde, 83 pp. [Available from Risø National Laboratory, 4000 Roskilde, Denmark.]

——, 1994: Cups, props and vanes. Risø National Laboratory Risø-R-766(EN), Roskilde, 36 pp. [Available from Risø National Laboratory, 4000 Roskilde, Denmark.]

——, D. H. Lenschow, P. Kirkegaard, and M. S. Courtney, 1989: The spectral velocity tensor for homogeneous boundary layer turbulence. *Bound.-Layer Meteor.,* **47,** 149–193.

MacCready, P. B., Jr., 1965: Dynamic response characteristics of meteorological sensors. *Bull. Amer. Meteor. Soc.,* **46,** 533–538.

——, 1966: Mean speed measurements in turbulence. *J. Appl. Meteor.,* **5,** 219–225.

Middleton, W. E. K., 1969: *Invention of the Meteorological Instruments.* The Johns Hopkins Press, 362 pp.

Panofsky, H. A., and J. A. Dutton, 1984: *Atmospheric Turbulence.* John Wiley and Sons, 397 pp.

Patterson, J., 1926: The cup anemometer. *Trans. Roy. Soc. Canada, Ser. III,* **20,** 1–54.

Sabinin, G., 1923: Les anémometers à moulinet et la mesure à leur aide de la vitesse réele du vent. *Bull. Moscow Inst. Space Phys.,* **1,** 196–214.

Schrenk, O., 1929: Über die Trägheitsfehler des Schalenkreuz-anemometer bei schwankender Windstärke. *Z. Tech. Phys.,* **10,** 57–66.

Stow, F. W., 1872: On large and small anemometers. *Quart. J. Meteor. Soc.,* **1,** 41–49.

Wyngaard, J. C., 1981: Cup, propeller, vane, and sonic anemometers in turbulence research. *Annu. Rev. Fluid Mech.,* **13,** 399–423.

——, and O. R. Coté, 1972: Cospectral similarity in the atmospheric surface layer. *Quart. J. Roy. Meteor. Soc.,* **98,** 590–603.

——, J. T. Bauman, and R. A. Lynch, 1974: Cup anemometer dynamics. *Proc. Flow, Its Measurements and Control in Science and Industry,* Pittsburgh, PA, Instrument Society of America, 10–14.