Bias on Wind-Speed Variance caused by Quantization and Sub-Interval Counting of the Cup-Anemometer Signal

Leif Kristensen & Ole Frost Hansen

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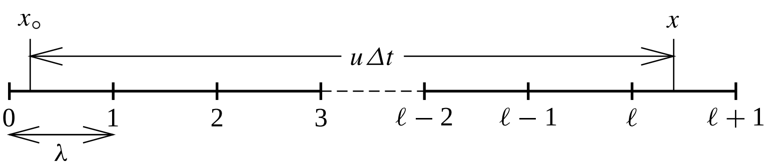
Short Presentation and Conclusion

Every time a column of air of length λ passes through the cup anemometer rotor a pulse is created. For the Risø model P2546, $\lambda = 0.62$ m and, neglecting in these considerations the bias or starting speed of $+0.27$ m s^{-1}, the count $N = 7000$ over the time $T = 10$ min $= 600$ s corresponds to a wind way of 4340 m and a mean-wind speed of about U =7.2 m s^{-1}. The counting does not in general start and stop at the arrival of a pulse and, consequently, the count may be off by 1 corresponding to an uncertainty in the mean-wind speed of 0.001 m s^{-1}. This is under all circumstances negligible. However, the pulses are also counted over much smaller time intervals Δt, typically 2 s, and from the $M = T/\Delta t = 300$ corresponding short *sub-interval* values of the wind speed the variance is calculated over the time T. In this case the counting uncertainty will in principle give extra variance. It it the purpose in this note to present an estimate of this variance bias. It can be obtained immediately by applying the so-called Sheppard's correction (Sheppard 1898, Wold 1934, Cramér 1946, Kristensen & Kirkegaard 1987). Here we will derive the result directly.



The count ℓ over the time Δt has a mean and a variance which is determined by

$$\langle \ell \rangle = \frac{1}{M}\sum_{m=0}^{M-1} \ell_m \tag{1}$$

and

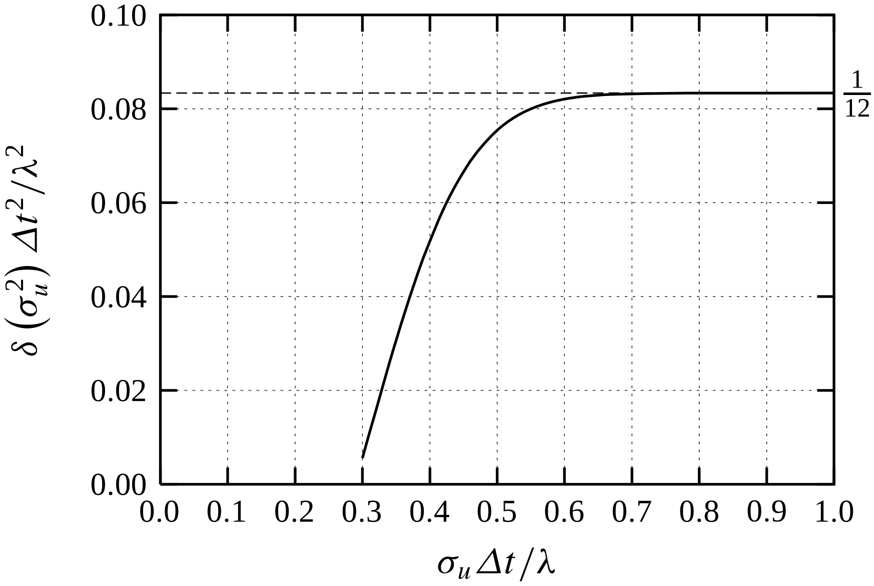
$$\sigma_\ell^2 = \langle (\ell - \langle \ell \rangle)^2 \rangle = \frac{1}{M}\sum_{m=0}^{M-1} (\ell_m - \langle \ell \rangle)^2, \tag{2}$$

where ℓ_m is count number m in the period in question.

There is the following relation between the variance, $\sigma_u^2 = \langle (u - U)^2 \rangle$ of u and σ_ℓ^2

$$\frac{\sigma_\ell^2 \lambda^2}{\Delta t^2} = \sigma_u^2 + \frac{\lambda^2}{12 \Delta t^2}. \tag{3}$$

Of course, this equation cannot be valid if the variance σ_u^2 is zero or very small. However, Kristensen & Kirkegaard (1987) found that (3) is accurate within 2.5% for a Gaussian process if $\sigma_u > \lambda/(2\Delta t)$. The correction is shown below



Equation (3) implies that for the Risø model P2546 the variance will be overestimated by the constant amount $\lambda^2/\Delta t^2/12 \approx 0.008$ m^2 s^{-2} if $\Delta t = 2$ s. The result (3) may be reformulated in terms of the standard deviation SD as follows

$$\mathrm{SD} \equiv \sqrt{\frac{\sigma_\ell^2 \lambda^2}{\Delta t^2}} = \sigma_u \sqrt{1 + \frac{1}{\sigma_u^2} \frac{\lambda^2}{12\, \Delta t^2}} \simeq \sigma_u + \frac{1}{\sigma_u} \frac{\lambda^2}{24\, \Delta t^2}. \tag{4}$$

For the Risø model P2546 we find that with $\Delta t = 2$ s and $\sigma_u = 1$ m s^{-1}, corresponding to the wind speed 10 m s^{-1} and the turbulence intensity 0.1, we get the bias correction SD $- \sigma_u = 0.004$ m s^{-1}.

In the following we will derive the result (3).

Derivation of Equation (3)

Obviously we have the simple relation

$$\langle(\ell - \langle\ell\rangle)^2\rangle = \langle\ell^2\rangle - \langle\ell\rangle^2 \tag{5}$$

and we proceed by determining $\langle\ell\rangle$ and $\langle\ell^2\rangle$ on basis of the probability density function for the wind speed u averaged over the time interval Δt. First we derive the probability $P[\ell]$ for the count ℓ from the probability density $\phi_\circ(y)$ for the wind way $y = u\Delta t$ which, since Δt is kept constant, is equal to the probability density function for u divided by Δt. Inspecting the sketch, we see that $P[\ell]$ is the probability that the beginning of the wind way $0 \le x_\circ \le \lambda$ falls in the first λ-interval and that the end $x = x_\circ + y$ falls in the ℓ'th λ-interval. In other words

$$P[\ell] = \int_0^\lambda \frac{\mathrm{d}x_\circ}{\lambda} \int_{\ell\lambda}^{(\ell+1)\lambda} \phi_\circ(x - x_\circ)\,\mathrm{d}x. \tag{6}$$

Introducing for convenience

$$\phi(x) = \int_0^\lambda \phi_\circ(x - x_\circ)\,\frac{\mathrm{d}x_\circ}{\lambda}, \tag{7}$$

we can rewrite (6) in the form

$$P[\ell] = \int_{\ell\lambda}^{(\ell+1)\lambda} \phi(x)\,\mathrm{d}x. \tag{8}$$

Since

$$\int_0^\infty \phi(x)\,\mathrm{d}x = \int_0^\lambda \frac{\mathrm{d}x_\circ}{\lambda} \int_0^\infty \phi_\circ(x - x_\circ)\,\mathrm{d}x = 1, \tag{9}$$

we see immediately that

$$\sum_{\ell=0}^{\infty} P[\ell] = 1. \tag{10}$$

The mean of ℓ and ℓ^2 become

$$\begin{aligned}\langle \ell \rangle &= \sum_{\ell=0}^{\infty} \ell P[\ell] = \sum_{\ell=0}^{\infty} \ell \int\limits_{\ell\lambda}^{(\ell+1)\lambda} \phi(x)\,\mathrm{d}x = \sum_{\ell=0}^{\infty} \ell \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x - \sum_{\ell=0}^{\infty} \ell \int\limits_{(\ell+1)\lambda}^{\infty} \phi(x)\,\mathrm{d}x \\ &= \sum_{\ell=1}^{\infty} \ell \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x - \sum_{\ell=1}^{\infty} (\ell-1) \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x = \sum_{\ell=0}^{\infty} \int\limits_{(\ell+1)\lambda}^{\infty} \phi(x)\,\mathrm{d}x \end{aligned} \tag{11}$$

and

$$\begin{aligned}\langle \ell^2 \rangle &= \sum_{\ell=0}^{\infty} \ell^2 P[\ell] = \sum_{\ell=0}^{\infty} \ell^2 \int\limits_{\ell\lambda}^{(\ell+1)\lambda} \phi(x)\,\mathrm{d}x = \sum_{\ell=0}^{\infty} \ell^2 \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x - \sum_{\ell=0}^{\infty} \ell^2 \int\limits_{(\ell+1)\lambda}^{\infty} \phi(x)\,\mathrm{d}x \\ &= \sum_{\ell=1}^{\infty} \ell^2 \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x - \sum_{\ell=1}^{\infty} (\ell-1)^2 \int\limits_{\ell\lambda}^{\infty} \phi(x)\,\mathrm{d}x = \sum_{\ell=0}^{\infty} (2\ell+1) \int\limits_{(\ell+1)\lambda}^{\infty} \phi(x)\,\mathrm{d}x. \end{aligned} \tag{12}$$

To proceed we need an approximate relation between infinite summations and infinite integrals. We start with the identity

$$\int\limits_{0}^{\infty} f(x)\,\mathrm{d}x = \sum_{\ell=0}^{\infty} \int\limits_{\ell\lambda}^{(\ell+1)\lambda} f(x)\,\mathrm{d}x, \tag{13}$$

where $f(x)$ can be any integrable function for which $f'(\infty) = 0$. Taylor expanding of $f(x)$ from the midpoint $(\ell + 1/2)\lambda$ yields

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell+1/2)\lambda)}{n!} (x - (\ell+1/2)\lambda)^n\,, \tag{14}$$

where $f^{(n)}(x)$ is the n'th derivative of $f(x)$. Thus

$$
\begin{aligned}
\int_{\ell\lambda}^{(\ell+1)\lambda} f(x)\,\mathrm{d}x &= \int_{\ell\lambda}^{(\ell+1)\lambda} \mathrm{d}x \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell+1/2)\lambda)}{n!}\,(x-(\ell+1/2)\lambda)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell+1/2)\lambda)}{n!} \int_{-\lambda/2}^{\lambda/2} s^n\,\mathrm{d}s = \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell+1/2)\lambda)}{(n+1)!} \left\{ \left(\frac{\lambda}{2}\right)^{n+1} - \left(-\frac{\lambda}{2}\right)^{n+1} \right\} \\
&= \sum_{n=0}^{\infty} \frac{f^{(2n)}((\ell+1/2)\lambda)}{2^{2n}(2n+1)!} \lambda^{2n+1}. \qquad (15)
\end{aligned}
$$

Inserting in (13), we obtain

$$
\int_0^{\infty} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2^{2n}(2n+1)!} \sum_{\ell=0}^{\infty} f^{(2n)}((\ell+1/2)\lambda). \qquad (16)
$$

This relation is one form of the Euler-Maclaurin sum formula. Since $f(x)$ can be any function it also applies to its second derivative $f''(x)$, i.e.

$$
\int_0^{\infty} f''(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2^{2n}(2n+1)!} \sum_{\ell=0}^{\infty} f^{(2n+2)}((\ell+1/2)\lambda). \qquad (17)
$$

Truncating (16) after the second term and (17) after the first, we have

$$
\int_0^{\infty} f(x)\,\mathrm{d}x \simeq \lambda \sum_{\ell=0}^{\infty} f((\ell+1/2)\lambda) + \frac{\lambda^3}{24} \sum_{\ell=0}^{\infty} f''((\ell+1/2)\lambda) \qquad (18)
$$

and

$$
\int_0^{\infty} f''(x)\,\mathrm{d}x \simeq \lambda \sum_{\ell=0}^{\infty} f''((\ell+1/2)\lambda). \qquad (19)
$$

Combining these two equations, we get

$$\int_0^\infty f(x)\,\mathrm{d}x \simeq \lambda \sum_{\ell=0}^{\infty} f((\ell+1/2)\lambda) + \frac{\lambda^2}{24}\int_0^\infty f''(x)\,\mathrm{d}x = \lambda \sum_{\ell=0}^{\infty} f((\ell+1/2)\lambda) - \frac{\lambda^2}{24} f'(0) \quad (20)$$

or

$$\sum_{\ell=0}^{\infty} f((\ell+1/2)\lambda) \simeq \frac{1}{\lambda}\int_0^\infty f(x)\,\mathrm{d}x + \frac{\lambda}{24} f'(0). \quad (21)$$

We can now determine $\langle \ell \rangle$ and $\langle \ell^2 \rangle$. We start with (11).

$$\begin{aligned}
\langle \ell \rangle &= \sum_{\ell=0}^{\infty} \underbrace{\int_{(\ell+1)\lambda}^{\infty} \phi(s)\,\mathrm{d}s}_{f((\ell+1/2)\lambda)} \simeq \frac{1}{\lambda}\int_0^\infty \mathrm{d}x \int_{x+\lambda/2}^{\infty} \phi(s)\,\mathrm{d}s - \frac{\lambda}{24}\phi(\lambda/2) \\
&= \frac{1}{\lambda}\int_0^\infty x\phi(x+\lambda/2)\,\mathrm{d}x - \frac{\lambda}{24}\phi(\lambda/2) \simeq \frac{\langle u \rangle \Delta t}{\lambda}. \quad (22)
\end{aligned}$$

$$\begin{aligned}
\langle \ell^2 \rangle &= \sum_{\ell=0}^{\infty} \underbrace{(2\ell+1) \int_{(\ell+1)\lambda}^{\infty} \phi(s)\,\mathrm{d}s}_{f((\ell+1/2)\lambda)} \simeq \frac{1}{\lambda^2}\int_0^\infty 2x\,\mathrm{d}x \int_{x+\lambda/2}^{\infty} \phi(s)\,\mathrm{d}s + \frac{1}{12}\int_{\lambda/2}^{\infty} \phi(s)\,\mathrm{d}s \\
&= \frac{1}{\lambda^2}\int_0^\infty x^2\phi(x+\lambda/2)\,\mathrm{d}x + \frac{1}{12}\int_{\lambda/2}^{\infty} \phi(s)\,\mathrm{d}s \simeq \frac{\langle u^2 \rangle \Delta t^2}{\lambda^2} + \frac{1}{12}. \quad (23)
\end{aligned}$$

The last two equations imply that the measured variance becomes

$$\langle (\ell - \langle \ell \rangle)^2 \lambda^2 \rangle \simeq \langle (u - \langle u \rangle)^2 \rangle \Delta t^2 + \frac{\lambda^2}{12}. \quad (24)$$

The Constant Wind-Speed Case

We have so far assumed that the wind-speed probability has a certain width corresponding to the atmospheric turbulence. However, in the very extreme case where the wind speed is constant there will be fluctuations of the recorded signal if the total wind way $\Lambda = u\Delta t$ differs from an integer number of one-pulse wind way λ. We must modify the derivation of the mean and the variance outlined in the previous section.

The pdf $\phi_\circ(y)$ for the constant wind way Λ is

$$\phi_\circ(y) = \delta(y - \Lambda), \tag{25}$$

where $\delta(x)$ is Dirac's delta function.

Corresponding to (6) and (7) we have

$$P[\ell] = \int_0^\lambda \frac{\mathrm{d}x_\circ}{\lambda} \int_{\ell\lambda}^{(\ell+1)\lambda} \delta(x - x_\circ - \Lambda)\,\mathrm{d}x \tag{26}$$

or

$$P[\ell] = \int_{\ell\lambda}^{(\ell+1)\lambda} \phi(x)\,\mathrm{d}x, \tag{27}$$

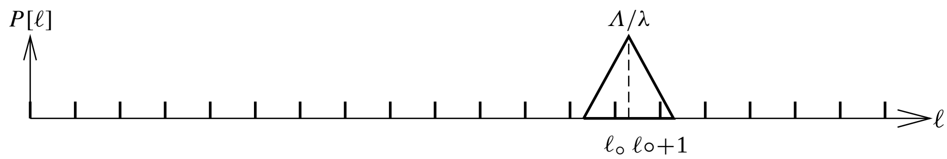
where

$$\phi(x) = \int_0^\lambda \delta(x - x_\circ - \Lambda)\,\frac{\mathrm{d}x_\circ}{\lambda} = \begin{cases} 1/\lambda, & \Lambda < x < \Lambda + \lambda \\ 0 & \text{elsewhere} \end{cases} . \tag{28}$$

Carrying out the integration (26) we get

$$P[\ell] = \begin{cases} 1 - |\ell - \Lambda/\lambda|\,, & (\Lambda/\lambda - 1) \le \ell < (\Lambda/\lambda + 1) \\ 0 & \text{elsewhere} \end{cases} . \tag{29}$$

A sketch of $P[\ell]$ is shown below.



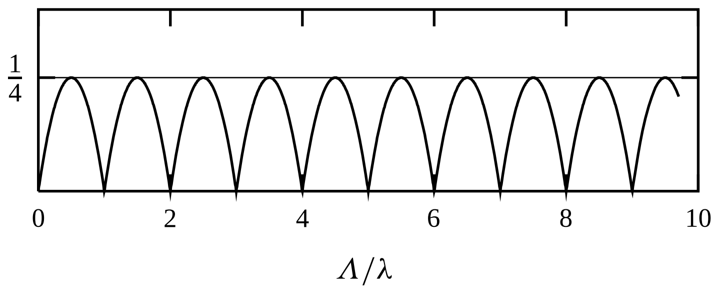
The formulation (29) must be understood as a function of integers ℓ. The implication is that, for a given value of Λ/λ, there are only two neighboring values of ℓ for which $P[\ell]$ is different from zero. Let the these integer values be $\ell_\circ$ and $\ell_\circ + 1$. Inspecting the sketch we see that $P[\ell_\circ] = \sqrt{3}\,(\ell_\circ + 1 - \Lambda/\lambda)$ and $P[\ell_\circ + 1] = \sqrt{3}\,(\Lambda/\lambda - \ell_\circ)$. We see that the norm is $P[\ell_\circ] + P[\ell_\circ + 1] = \sqrt{3}$. Taking this normalization into account we get

$$m[\Lambda/\lambda, \ell_\circ] = \ell_\circ\,(\ell_\circ + 1 - \Lambda/\lambda) + (\ell_\circ + 1)\,(\Lambda/\lambda - \ell_\circ) = \Lambda/\lambda \tag{30}$$

for the mean and

$$\begin{aligned}\sigma_\circ^2(\Lambda/\lambda, \ell_\circ) &= \ell_\circ^2\,(\ell_\circ + 1 - \Lambda/\lambda) + (\ell_\circ + 1^2)\,(\Lambda/\lambda - \ell_\circ)\\ &= (\Lambda/\lambda - \ell_\circ)(\ell_\circ + 1 - \Lambda/\lambda)\end{aligned} \tag{31}$$

for the variance, which is shown below.



We see that the argument $\ell_\circ$ is superfluous in (30) and (31) and we may repeat the last equation in the form

$$\sigma_\circ^2(\Lambda/\lambda) = (\Lambda/\lambda - \lfloor\Lambda/\lambda\rfloor)(1 - (\Lambda/\lambda - \lfloor\Lambda/\lambda\rfloor)), \tag{32}$$

where $\lfloor\Lambda/\lambda\rfloor$ is the largest integer smaller than or equal to Λ/λ (“Floor” or “Entier” in computer language).

References

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